Mean Field Theory in Networks

Jiatu Shi

Outline

- Introduction: an example
- Mean field theory
- Topology of networks
- The correlation of the theorys
- Summary
- Reference

Introduction:an example

Graph model plays an important role in social network analysis.

Average distance is the essential topology index of a graph.

six degrees of separation[1]

Introduction:an example

How to calculate the average distance of an undirected and unweighted network?

$$L = \frac{2}{n(n-1)} \sum_{i \neq j} (d(v_i, v_j))$$

It seems easy.

But how about the complexity?

Introduction:an example

- Dijkstra or Floyd? ------- $O(n^3) \rightarrow O(n^2 logn)$ exact but slow.
- BFS? ------- $O(n^2 + ne) \rightarrow O(n^2)$ faster...
- What if $\mathrm{n} \approx 10^9$

  $O(n^2)$ will not be allowed!
- How about approximation algorithm?

  If I select just a few nodes, 500, 5000, 50000?

  Hmm…It will be $O(cn)$!

Introduction:an example

- Shortage of Approximation algorithm is obvious: The accuracy.
- Then I play a game:

  3 real network data[2]

  nearly BA degrees distribution

  #nodes $\approx$ 80,000 , 200,000 , 800,000

  As I randomly chose more than nearly 1000 of the total nodes

  ε is less than 0.5%

Introduction:an example

- But suddenly I had read such a paper[3], that the average distance of a BA network can be estimated from #nodes N and #edges M:

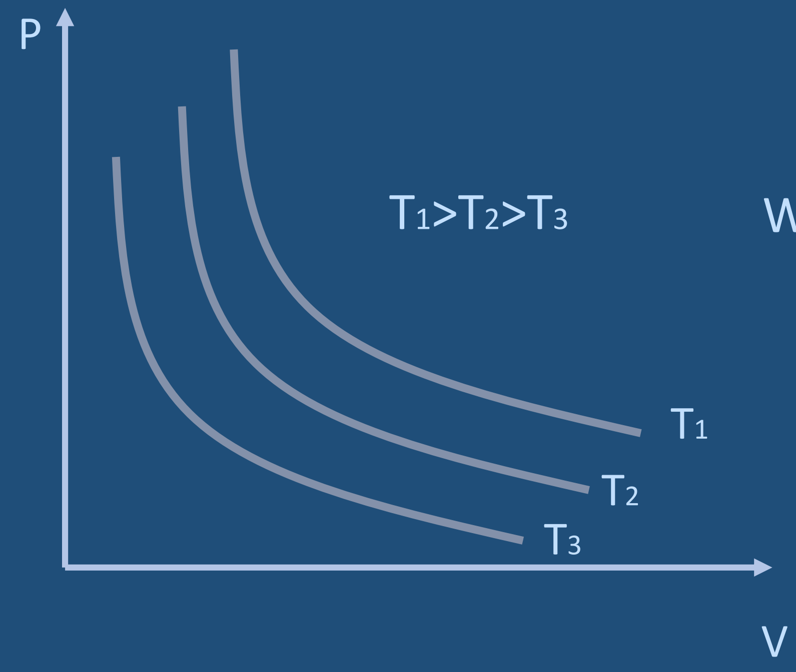
$$L \simeq \frac{lnN - ln\frac{m}{2} - 1 - \gamma}{lnlnN + ln\frac{m}{2}} + \frac{3}{2}$$

$\gamma \simeq 0.5772$ Euler constant

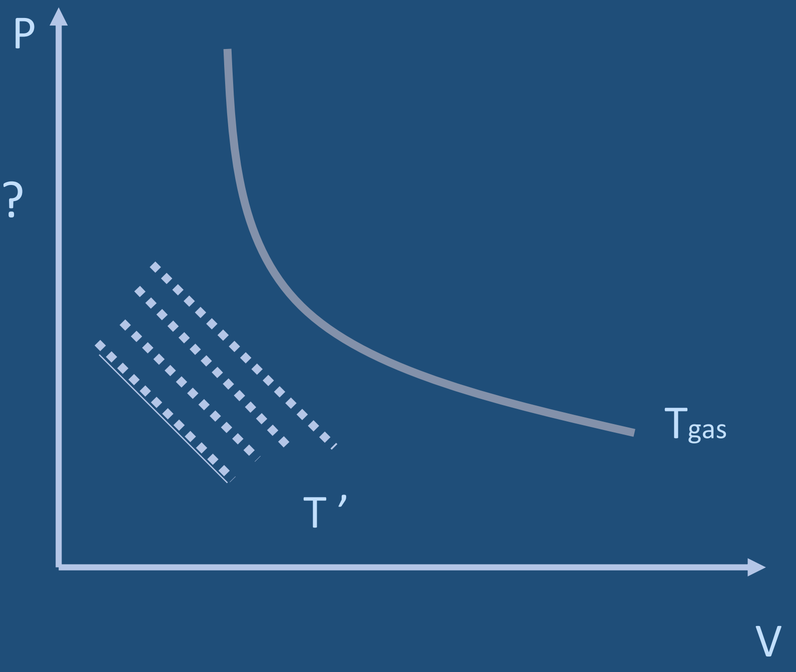
Amazing…

Mean field theory

- Ideal gas state equation: $PV = nRT = nkT$

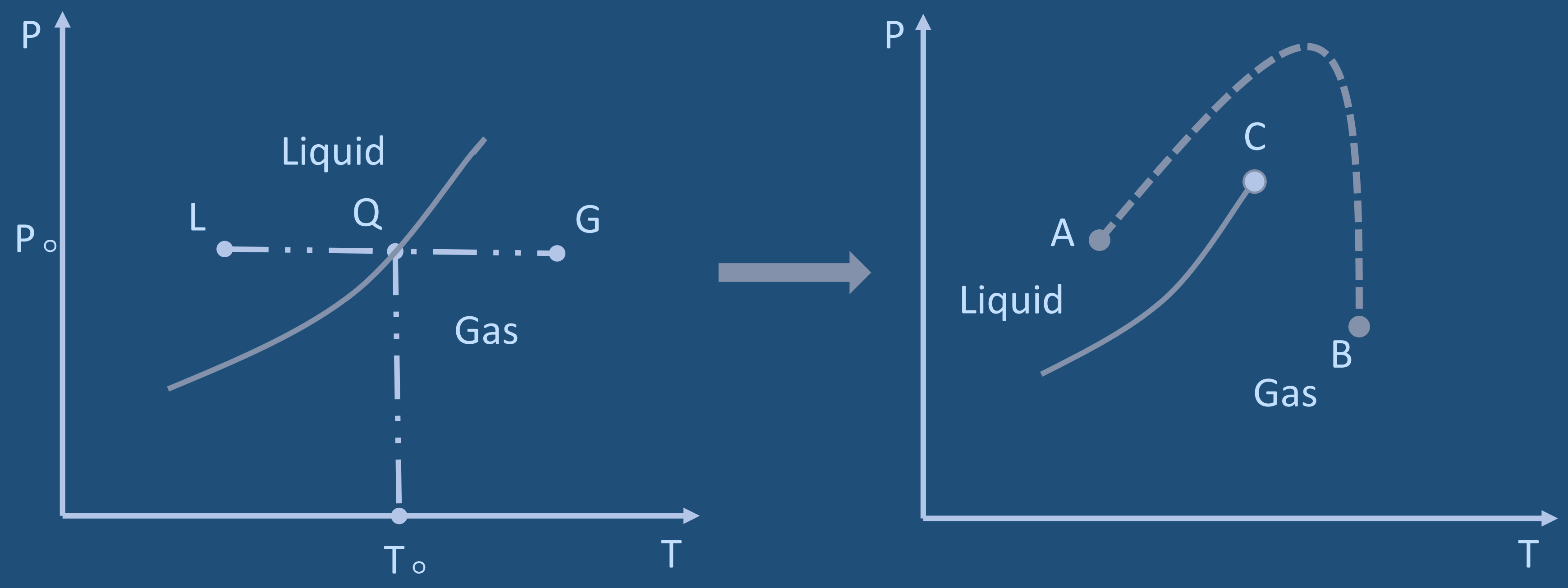


What is the Case : $T' \ll T_{gas}$?



Mean field theory

- One of the simplest phase transition: gas and liquid.[4]
- Is phase transition point limited? Of course.



Mean field theory

- What's the paradox at the point C?

compressibility increased

↓

scatter enhanced

↓

opalesce

Mean field theory

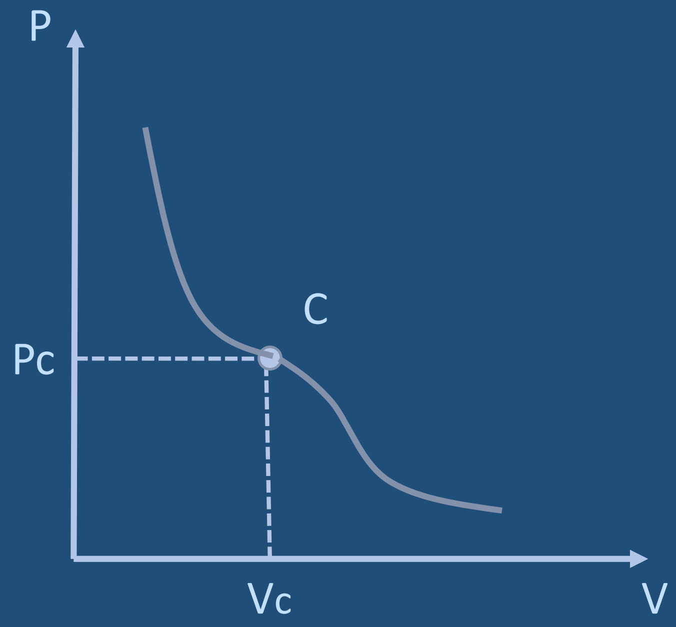
- A revised theory : Van der Waals equation

1. The moleculen have a certain volume. $V \rightarrow V - Nb$
2. The moleculen have attraction when mutual distant. $P \rightarrow P + (\frac{N}{V})^2 a$

$$\left(P + \left(\frac{N}{V}\right)^2 a\right)(V - Nb) = NkT$$

Mean field theory

Van der Waals equation



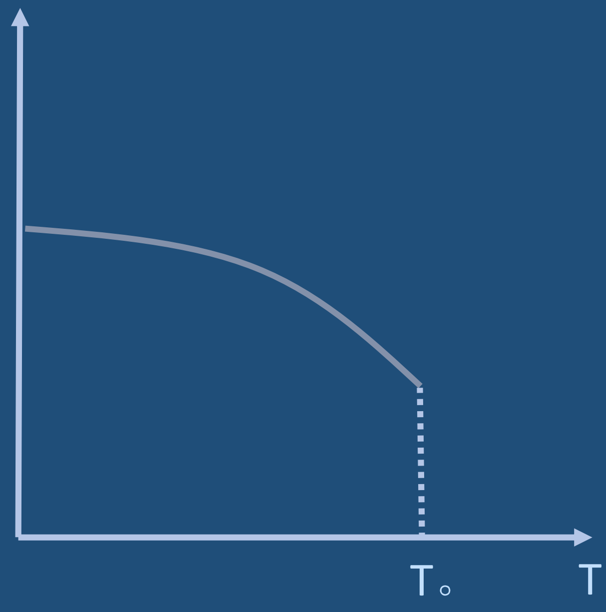
$$\begin{cases} \left(\frac{\partial P}{\partial V}\right)_T = 0 \\ \left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0 \end{cases} \Longrightarrow \begin{cases} T_c = \frac{8a}{27bk} \\ P_c = \frac{a}{27b^2} \\ V_c = 3Nb \end{cases}$$

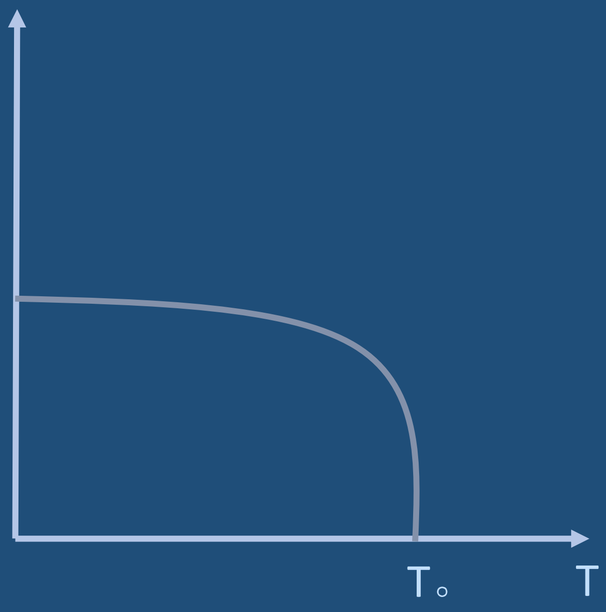
Let $t' = \frac{T}{T_c}$ $\quad p' = \frac{P}{P_c}$ $\quad v' = \frac{V}{V_c}$

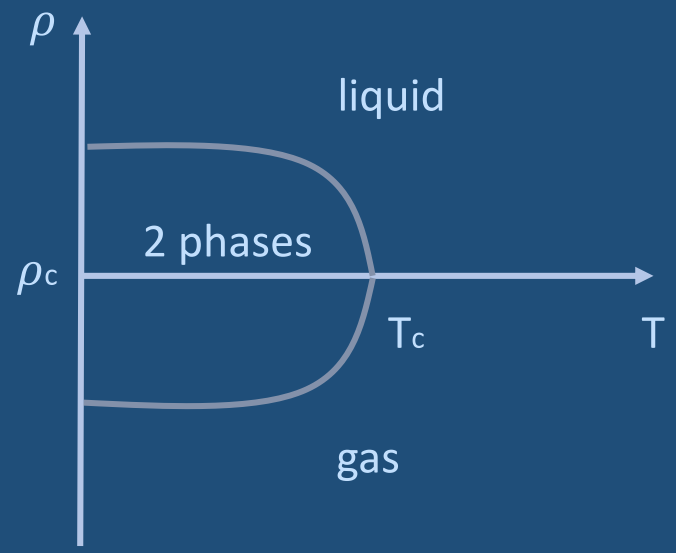
$$\left(P + \left(\frac{N}{V}\right)^2 a\right)(V - Nb) = NkT \Longrightarrow \left(p' + \frac{3}{v'^2}\right)(3v' - 1) = 8t'$$

Mean field theory

Van der Waals equation







$$\frac{\rho_l - \rho_g}{\rho_c} \propto \left(\frac{T_c - T}{T}\right)^{\beta} \qquad \beta = \frac{1}{2}$$

Topology of networks[6]

- Degree distribution

$p_k = \textit{The probability of the chosen node whose degree is k}$

- Diameter and average distance

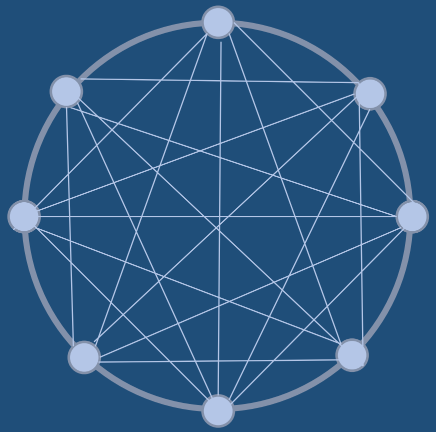
$D = max\, d(v_i, v_j)\,, i \neq j$ $\qquad L = \frac{2}{n(n-1)} \sum_{i \neq j} (d(v_i, v_j))$

- Clustering coefficient

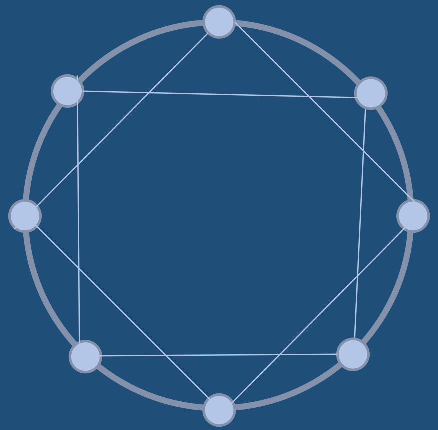
$C_i = \frac{2E_i}{k_i(k_i - 1)}$ $\qquad E_i = \#edges\ of\ node_i's\ neighbors$

Topology of networks[6]

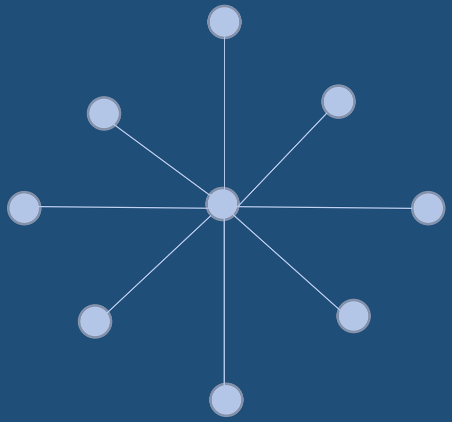
- Regular network



$$D = 1$$
$$C = 1$$



$$D = \frac{3N\binom{K/2}{2}}{N\binom{K}{2}} = \frac{3(K-2)}{4(K-1)}$$
$$C = 1$$

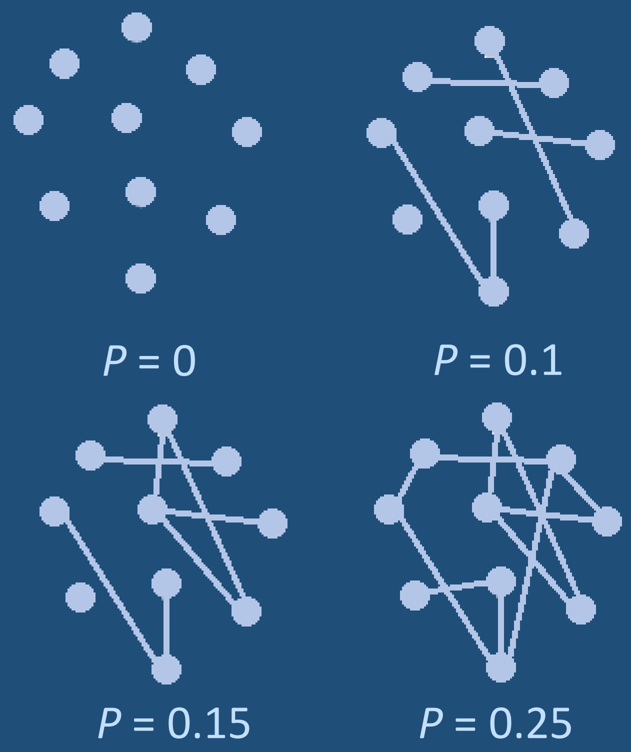


$$D = 2 - \frac{2(N-1)}{N(N-1)} \rightarrow 2 (N \rightarrow \infty)$$
$$C = 0$$

Topology of networks[6]

- ER random network

  each pair of nodes has a certain probability: *p, to be connected*



$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

Notice that N is large and p is small $\binom{N-1}{k} \approx \frac{(N-1)^k}{k!}$

$$\langle k \rangle = p(N-1)$$

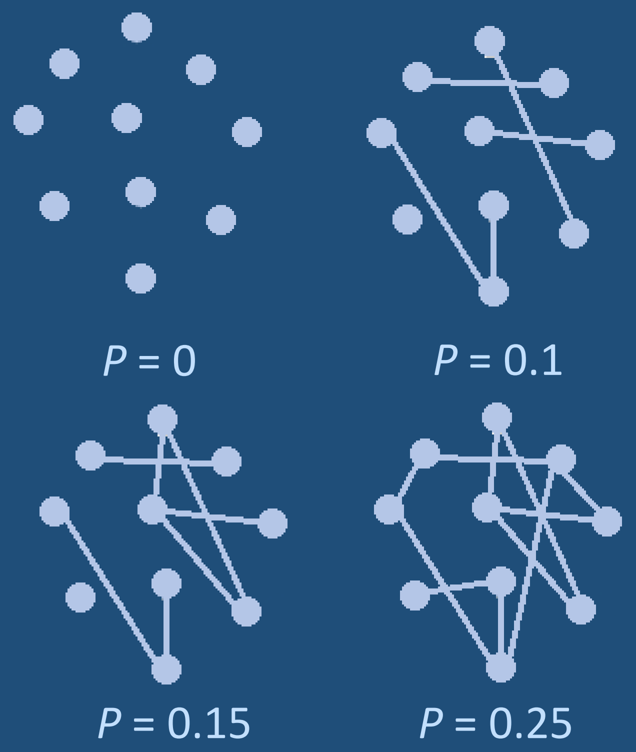
$\Longrightarrow$ $\ln[(1-p)^{N-1-k}] \approx -\frac{\langle k \rangle (N-1-k)}{(N-1)} \approx -\langle k \rangle$

$\Longrightarrow$ $P(k) \approx \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$ Possion random graph

Topology of networks[6]

- ER random network

  each pair of nodes has a certain probability: *p, to be connected*



$$C = p = \frac{\langle k \rangle}{N-1} \rightarrow 0$$

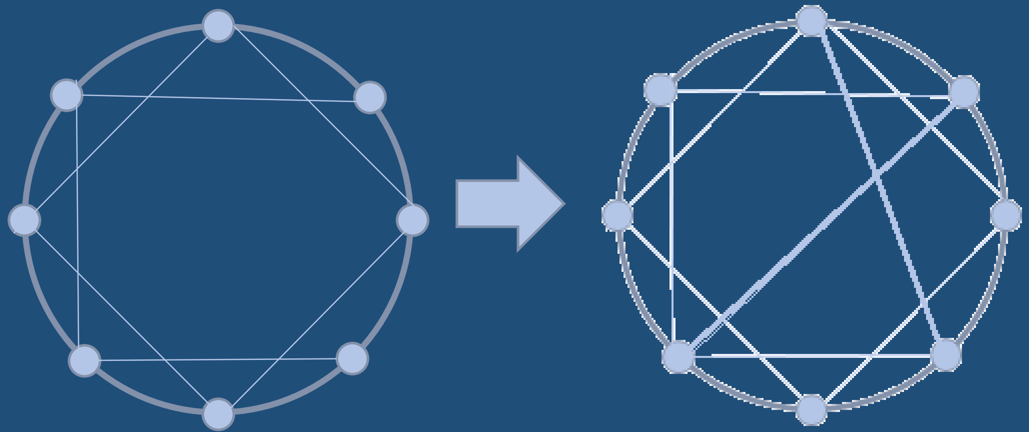
$$C\ is\ small\ \rightarrow N \sim \langle k \rangle^{D}$$

$$L \leq D \sim \frac{lnN}{ln\langle k \rangle}$$

Topology of networks[6]

- WS / NW small-world network

  each pair of nodes has a certain probability: *p, **add an edge***



$$C = \frac{\frac{3}{4}NK(\frac{1}{2}K-1)}{\frac{1}{2}NK(K-1)+NK^2p+\frac{1}{2}NK^2p^2}$$

$$= \frac{3(K-2)}{4(K-1)+4Kp(p+2)}$$

$$P(k) = \binom{N-1}{k-K}(\frac{Kp}{N-1})^{k-K}(1-\frac{Kp}{N-1})^{N-1-k+K}$$

$$\approx \frac{(Kp)^{k-K}}{(k-K)!}e^{-Kp}$$

The correlation of the theorys

- How about the L in the small-world network?

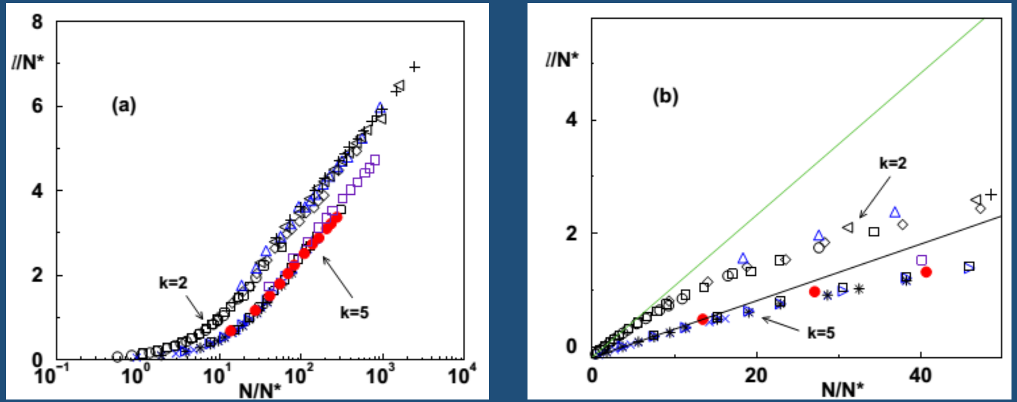
  It is a pity that the L can't be exactly parsed by expression.

  But there is an nearly accurate solution by mean field theory.[7]

$$L = \frac{N}{K} f(NKp)$$

$$f(x) = \frac{2}{\sqrt{x^2 + 4x}} arctanh \sqrt{\frac{x}{x+4}}$$

$$\text{arctanh}(x) = \frac{1}{2} ln \frac{1+x}{1-x}$$



The correlation of the theorys

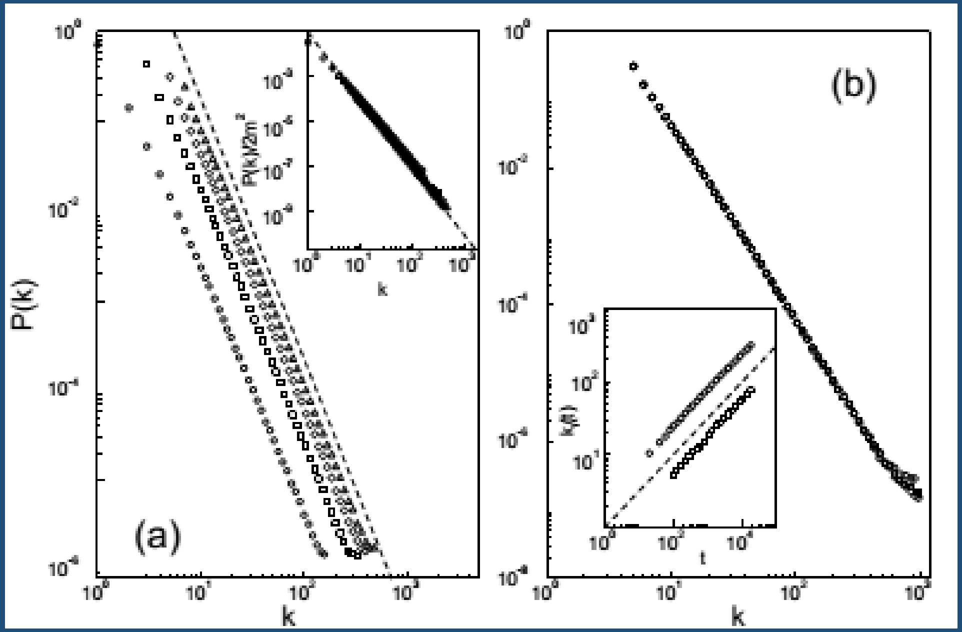
- BA network
- The network begins with an initial connected network of m_0 nodes.
- New nodes are added to the network one at a time. Each new node is connected to existing nodes with a probability that is proportional to the number of links that the existing nodes already have.

$$p_i = \frac{k_i}{\sum k_j}$$

The correlation of the theorys

- BA network

$$\frac{\partial k_i}{\partial t} = m\frac{k_i}{\sum k_j} = \frac{k_i}{2t} \qquad k_i(t) = m(\frac{t}{t_i})^{\beta} \qquad \beta = 0.5$$



$$P(k_i(t) < k) = P(t_i > \frac{m^{1/\beta}t}{k^{1/\beta}}) \qquad P(t_i) = \frac{1}{m_0 + t}$$

$$= 1 - \frac{m^{1/\beta}t}{k^{1/\beta}(t + m_0)}$$

$$P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{2m^{1/\beta}t}{m_0 + t}\frac{1}{k^{1/\beta+1}} \sim 2m^{1/\beta}k^{-\gamma}, with\ \gamma = \frac{1}{\beta} + 1 = 3$$

Summary

1. Studying the behavior of large and complex stochastic models by studying a simpler model.
2. Arose primarily in the field of statistical mechanics, but has more recently been applied in graphical models theory, artificial intelligence and elsewhere.
3. A zero-dimensional model, existing errors comparing with the accurate experiment. The high-order approximation being needed.

Reference

[1] Guare J. Six degrees of separation: A play[M]. Random House LLC, 1990.

[2] http://konect.uni-koblenz.de/networks/

[3] Fronczak A, Fronczak P, Hołyst J A. Average path length in random networks[J]. Physical Review E, 2004, 70(5): 056110.

[4]于渌，郝柏林，陈晓松. 边缘奇迹：相变与临界现象[M]. 北京：科学出版社，2005.7

[5] https://www.youtube.com/watch?v=cSliO89x7UU

[6] 汪小帆，李翔，陈关荣. 网络科学导论 [M]. 北京：高等教育出版社，2012.4

[7] Albert R, Barabási A L. Statistical mechanics of complex networks[J]. Reviews of modern physics, 2002, 74(1): 47.